DISCRETIZATION ERRORS IN FINITE ELEMENT ANALYSIS OF NON-LINEAR 2-D STRUCTURAL PROBLEMS WITH LARGE STRAINS AND PLASTICITY

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Abstract: A detailed study of discretization error estimators is proposed in this article. These estimators are essential to control the results obtained in a numerical finite element simulation of 2D structural problems with large strains and plasticity. Due to the nonlinearity of the analysis, not only the finite element mesh quality but also the time discretization accomplishment and the equation equilibrium error, need to be controlled. The developed estimators are flux projection class estimators based on the strain energy density generated by external loads. Some examples will show the performance of the proposed estimators.

1.0 INTRODUCTION

Recently, several references concerning mesh optimization have been proposed for structural dynamic problems [1], geometrically non-linear problems [2], elastoplasticity [3,4,5], frictional contact problems [6] or numerical simulation of forming processes [7,8]. However, nowadays there are only a few proposals on this subject concerning problems with large strains, plasticity and contact. This shortage is caused, mainly, by the lack of knowledge on the convergency properties of the finite element results of these problems [9], and the existence of new error sources. When a mathematical model is generated from the real physical system some simplificative hypothesis are assumed causing the first error of the mathematical model. When the physic laws that guide the problem are applied, the equations that determine the mechanical system behavior are obtained. The integration of these equations in complex continuous domains is necessary as a first step of its discretization, appearing then a new error: the discretization error. In the case of non-linear problems different components can be distinguished in this error. The discretization error can be separated into the spatial discretization error, time discretization error and the equation solving error. The first one is caused by the partitioning of a continuum into finite elements, and the second one appears because the equilibrium is established with a finite number of time steps. Both errors exist in problems, quasi-static or dynamic, that depend on load variation. The third error is produced due to the use of an implicit procedure to solve the equilibrium equations. This procedure implies an iterative loop obtaining only an approximate equilibrium of the system for each load step. This error, as the spatial discretization one, appears in every non-linear problem with a geometrical or material non-linearity. The error estimators suggested in this paper are flux projection class; hence a smoothing procedure [10] on the parameter employed to calculate the error is required. The use of the smoothing routine is proposed to obtain improved functions of the variables involved in the equilibrium equations solving process in order to improve the convergency of the solving sequence [11].

This article is focused on the spatial discretization error and the time discretization error. In fact, the component of the error caused by the imbalance of the equation system is presumed negligible [12] because the convergency criteria used is very restrictive.

2.0 ERROR ESTIMATORS

One of the most common methods employed to obtain useful estimators for non-linear problems is a generalization of estimators developed for linear problems. In this case a flux projection class estimator has been adapted to be used in elastic linear problems [13]. The estimator makes use of the strain energy density function in order to calculate the error energy norm.

First, for non-linear problems, the strain power function P_d generated in the body by the external loads is calculated:

$$\mathrm{P_d} = \int_V \sigma : \dot{\varepsilon} \; dv \tag{1}$$

Where σ is the Cauchy stress tensor, $\dot{\varepsilon}$ is the strain rate Eulerian tensor, and *V* the volume of the body. The scalar product of the above tensors $(\sigma : \dot{\varepsilon})$ is the strain power density. The difference between the exact strain power density: $(\sigma : \dot{\varepsilon})^{\sim}$ and the strain power density obtained directly using the finite element method): (marked with the symbol ~ from now on) has been chosen as the error *E*, of the finite element analysis solution in a point *M (x,y)* of the model and in the time *t* of the load sequence.

$$\mathrm{E(M,t) = E(x,y,t)} = (\sigma : \dot{\varepsilon}) - (\sigma : \dot{\varepsilon})^{\sim} \tag{2}$$

The two error components that center our interest are those related to the spatial and the time discretization. These error components are directly related with the element size and the time step size respectively. If both components are considered simultaneously a global error estimator is defined[14]. However, if the spatial and temporal components are evaluated separately, the spatial and the temporal discretization errors are obtained. In the following sections the theoretical definition and the incremental development of these two error estimators are presented.

2.1 Spatial discretization error

At first there is no possibility of obtaining the exact strain power density so it is necessary to estimate it. In the case of the spatial estimator, this estimation would be an "spatial improved value" (marked *s). This improved value is obtained using a function smoothing procedure involving only spatial parameters. So in the spatial discretization estimator the error is computed by:

$$\mathrm{E(x,y,t)} = (\sigma : \dot{\varepsilon})^{*_s} - (\sigma : \dot{\varepsilon})^{\sim} \tag{3}$$

Where $(\sigma : \dot{\varepsilon})^{*s}$ is the spatial improved field of the strain power density obtained using F.E.M., $(\sigma : \dot{\varepsilon})^{\sim}$. The accumulated value ${}^{\tau}E$, in one point of the model is calculated by integration of the absolute value in equation (3) from the initial moment when the load is applied to the time τ.

$$^{\tau}\mathrm{E} = \int_0^{\tau} \left| \mathrm{E}(\mathrm{x},\mathrm{y},\mathrm{t}) \right| dt \tag{4}$$

The error indicator $^{\tau}E_{v_e}$, is defined by integration of the error density in an element *e* of the mesh, being v_e the element *e* domain.

$$^{\tau}\mathrm{E}_{v_e} = \int_{v_e} \int_0^{\tau} \left| \mathrm{E}(\mathrm{x},\mathrm{y},\mathrm{t}) \right| dt dv \tag{5}$$

If the complete load sequence is integrated, i.e. from time *0* to time t_n, E_{v_e} will be obtained.

$$\mathrm{E}_{v_e} = \int_{v_e} \int_0^{\mathrm{t_n}} \left| \mathrm{E}(\mathrm{x},\mathrm{y},\mathrm{t}) \right| dt dv \tag{6}$$

If the purpose is the study of the error along the whole domain, the error estimator for the complete model *E* is calculated:

$$\mathrm{E} = \int_{\mathrm{V}} \int_0^{\mathrm{t_n}} \left| \mathrm{E}(\mathrm{x},\mathrm{y},\mathrm{t}) \right| dt dv \tag{7}$$

2.2 Time discretization error

Frequently, to achieve the development of efficient adaptive techniques it is not enough to check the element size in a finite element mesh. In the field of non-linear problems it is also necessary to control the time step size via an extra error component called time discretization error. This component is originated by the division of time in discrete intervals. This error component, as the spatial one, appears in analyses that have dependence on load history. Therefore it is interesting to try to separate the error in two parts: spatial and temporal. For this purpose a time error estimator is proposed in order to evaluate the error quota caused only by the time discretization. As in other estimators, the exact value of the strain power density is initially unknown. To estimate this function, a smoothing algorithm for the time parameter is used (signed $*t$). Hence, the time discretization error of the strain power density ⌊, in a point M *(x, y)* of the model and in the time *t*, is evaluated by:

$$\xi(x,y,t) = (\sigma : \dot{\varepsilon})^{*t} - (\sigma : \dot{\varepsilon})^{\sim} \tag{8}$$

The time error for the element is obtained by integration of the expression (8) along the element *e*:

$$\xi_{v_e} = \int_{v_e} \left| \xi(x,y,t) \right| dv \tag{9}$$

If, in addition to the integration in the complete domain, the process is integrated until time t_n, the error estimator ξ is determined:

$$\xi = \int_0^{t_n} \int_V \left| \xi(x,y,t) \right| dv dt \tag{10}$$

3.0 INCREMENTAL IMPLEMENTATION

In this section the incremental implementation for the error estimators is presented, intending to show the practical methods employed to calculate the discretization error components.

3.1 Spatial error

Along a time interval Δt, the spatial error density varies on each point of the model $\Delta E = {}^{t+\Delta t}_{t}E$:

$$ {}^{t+\Delta t}_{t}E = \int_{t}^{t+\Delta t} E\,dt = \int_{t}^{t+\Delta t} \left| (\sigma : \dot{\varepsilon})^{*_s} - (\sigma : \dot{\varepsilon})^{\sim} \right| dt \qquad (11)$$

or,

$$ {}^{t+\Delta t}_{t}E = \int_{t}^{t+\Delta t} \left| (\sigma : d\varepsilon)^{*_s} - (\sigma : d\varepsilon)^{\sim} \right| \qquad (12)$$

Where, $(\sigma : d\varepsilon)$ is the strain work differential of the body in a point *M (x, y)* during the time differential *dt*. It can also be expressed as:

$$ d\varphi = \sigma : d\varepsilon \qquad (13)$$

and $(\sigma : d\varepsilon)^{*s}$ is an improved strain work differential with respect to the spatial variable in a point *M(x, y)* of the body during the time differential *dt*.

Adding the relation (13) to (12), the spatial error gain is expressed as a function of the improved and discrete work density differentials.

$$ {}^{t+\Delta t}_{t}E = \int_{t}^{t+\Delta t} \left| (d\varphi)^{*_s} - (d\varphi)^{\sim} \right| \qquad (14)$$

Using a linear approach, the spatial error density gain can be expressed as:

$$ {}^{t+\Delta t}_{t}E = \left| \left({}^{t+\Delta t}_{t}\varphi\right)^{*s} - \left({}^{t+\Delta t}_{t}\varphi\right)^{\sim} \right| \qquad (15)$$

being $\left({}^{t+\Delta t}_{t}\varphi\right)^{*s}$ the improved value of the strain work density gain with respect to the spatial variable exclusively.

The spatial error indicator, ${}^{t+\Delta t}_{t}E_{v_e}$ during a time interval Δt, is defined as the integral of expression (15) on the domain v_e, of the element *e*.

$$ {}^{t+\Delta t}_{t}E_{v_e} = \int_{v_e} \left| \left({}^{t+\Delta t}_{t}\varphi\right)^{*_s} - \left({}^{t+\Delta t}_{t}\varphi\right)^{\sim} \right| dv \qquad (16)$$

For the complete load sequence [0, t_n] in an element *e*, the compiled spatial error ${}^{t_n}_{o}E_{v_e} = E_{v_e}$ is calculated for the mentioned element.

$$ E_{v_e} = \sum_{i=0}^{n-1} {}^{t_i+\Delta t_i}_{t_i}E_{v_e} \qquad (17)$$

Therefore the accumulated spatial error of the entire model for the complete load sequence E, is:

$$E = \sum_{e=1}^{N} E_{v_e} \tag{18}$$

Where N is the number of elements of the F.E. mesh.

3.1 Time error

The time error density variation $\Delta\xi = {}^{t+\Delta t}_{t}\xi$ on each point of the model during a time step Δt, can be expressed in the form:

$${}^{t+\Delta t}_{t}\xi = \int_t^{t+\Delta t} \xi \, dt = \int_t^{t+\Delta t} \left| (\sigma : \dot{\varepsilon})^{*_t} - (\sigma : \dot{\varepsilon})^{\sim} \right| dt \tag{19}$$

Which can also be given as a function of the strain work differentials $(\sigma : d\varepsilon)^{*_t}$ and $(\sigma : d\varepsilon)^{\sim}$. The former one has been improved only with respect to the time variable and the latter has been calculated directly using F.E.

$${}^{t+\Delta t}_{t}\xi = \int_t^{t+\Delta t} \left| (\sigma : d\varepsilon)^{*_t} - (\sigma : d\varepsilon)^{\sim} \right| \tag{20}$$

The time error density gain (20) can also be written as a function of the improved and discrete work density differentials $(d\varphi)^{*_t}$ and $(d\varphi)^{\sim}$.

$${}^{t+\Delta t}_{t}\xi = \int_t^{t+\Delta t} \left| (d\varphi)^{*_t} - (d\varphi)^{\sim} \right| \tag{21}$$

Using the same linear approximations employed for the spatial error, the time error density gain is:

$${}^{t+\Delta t}_{t}\xi = \left| \left({}^{t+\Delta t}_{t}\varphi\right)^{*t} - \left({}^{t+\Delta t}_{t}\varphi\right)^{\sim} \right| \tag{22}$$

where $\left({}^{t+\Delta t}_{t}\varphi\right)^{*t}$ is the improved value of the strain work density gain using only the temporal variable.

During this period Δt, the time error gain ${}^{t+\Delta t}_{t}\xi_{v_e}$ is evaluated as the integral of (21) through the domain v_e, defined by the element e

$${}^{t+\Delta t}_{t}\xi_{v_e} = \int_{v_e} \left| \left({}^{t+\Delta t}_{t}\varphi\right)^{*_t} - \left({}^{t+\Delta t}_{t}\varphi\right)^{\sim} \right| dv \tag{23}$$

For the complete load sequence the accumulated time error indicator of element e, is obtained as:

$$ {}_{t_0}^{t_n}\xi_{v_e} = \xi_{v_e} = \sum_{i=0}^{n-1} {}_{t_i}^{t_i+\Delta t_i}\xi_{v_e} \tag{24} $$

and the accumulated time error of the entire model for the complete process ξ will be:

$$ {}_{t_0}^{t_n}\xi_V = \xi = \sum_{e=1}^{N} \xi_{v_e} \tag{25} $$

4.0 NUMERICAL EXAMPLE

The procedures described in this paper will be applied to the example of Figure 1 and numerical results of this particular case will be obtained. Figure 1 defines the geometry, dimensions and applied loads of the problem. This particular example involves a rectangular plate with a central hole subjected to uniform axial stress, σ, on both sides.

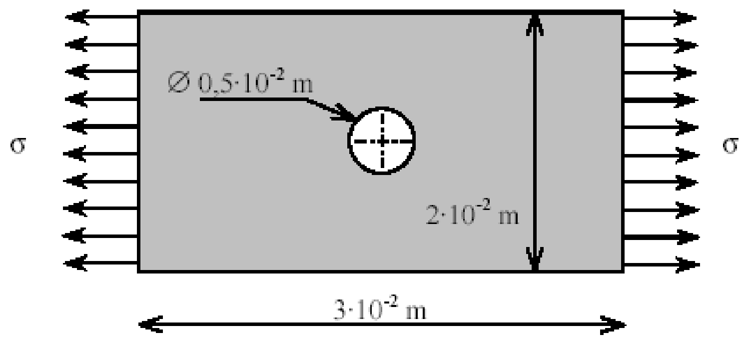

Figure 1. Problem definition.

Thanks to geometric and load symmetry of the problem it is enough to analyze only one fourth of the complete model. Figures 2a, 2b and 2c present different finite element meshes, with their correspondent degrees of freedom, to be analyzed.

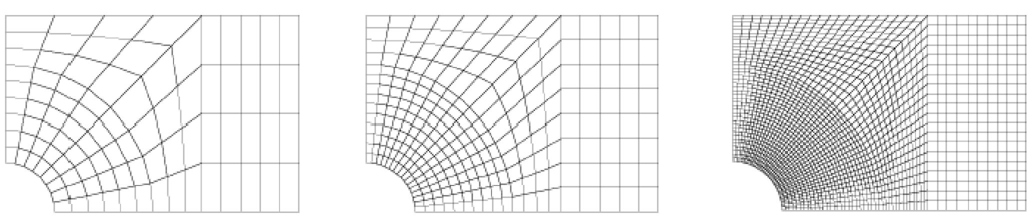

Figure 2a. Plate A - 230 dof. Figure 3b. Plate B - 600 dof. Figure 3c. Plate C - 3782 dof.

The methodology selected for the study of both discretization error estimators consists of different analysis associated to the distinct material complishment: linear elastic, elastoplastic, with small strains and elastoplastic with large strains.

4.1 Linear elastic analysis

In order to verify that the proposed procedure is applicable to a linear static problem, a linear static analysis has been carried out. The error estimator[14] in this case, is a particularisation of the proposed estimator. Obviously, in this problem the time discretization error component does no exist. The applied load is 40 MPa. The accumulated spatial discretization error during the process, for every model, is shown in Figure 3. It can be seen the adjustment of those curves to parabolic functions.

4.2 Elastoplastic analysis – small strains
The first non-linear calculation corresponds to the case of elastoplasticity with small strains and under an applied load σ of 150 MPa.

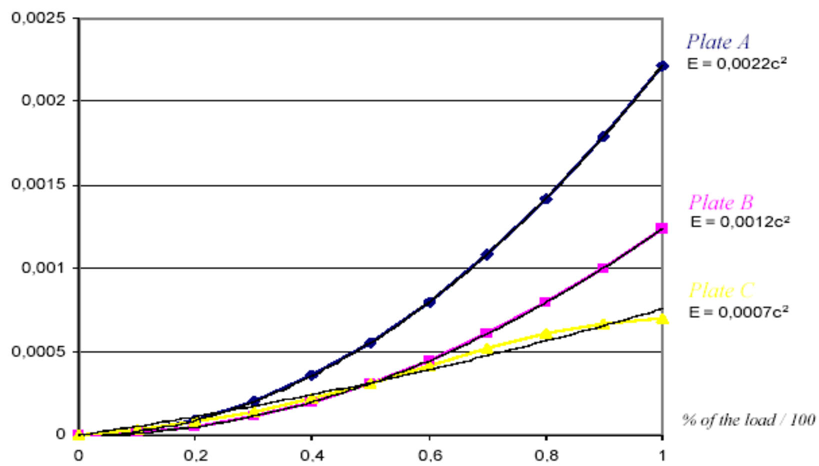

Figure 3. Spatial discretization error E ($x10^{-1}$J)– linear elastic case.

The material behavior law used on this simulation is bilinear with a Young module of $2{,}1\cdot10^5$ MPa, yield stress equal to $2{,}1\cdot10^5$ MPa and a plastic zone slope of 1000 MPa.

In order to solve these non-linear problems, with small or large strains, the algorithm presented in this paper includes an updated Lagrangian formulation to solve the equilibrium equations using the Jaumman's stress variation tensor. The plasticity law is based on Von Mises yield criteria with isotropic hardening and associative flux rule.

In the table below, table 1, the maximum displacement, stress and plastic strains are shown for each of the three meshes. In the last column the plastification order can be seen. These values situate this analysis in the small strain type group.

Mesh	Maxim. displacement. ($x\ 10^{-3}$ m)	Maxim. Von Mises Stress. ($x\ 10^5$ Pa)	Maximum plastic strain. (%)
Plate A	$4{,}08\cdot10^{-2}$	1857	3,62
Plate B	$4{,}51\cdot10^{-2}$	1934	4,03
Plate C	$4{,}66\cdot10^{-2}$	1944	5,02

Table 1. Displacements - Stresses – Plastic strains.

4.2.1 Spatial error estimator
Different calculations have been done for each model. In each of them only the load increment has been modified. The values for the load increment are 4%, 3%, 2% and 1% of

the total applied load. The values of each of the spatial discretization errors obtained for each mesh considering the four load increments can be seen below.

In figure (5) can be seen how the spatial error is weakly affected by the load increment variation but it is very dependent on the element size modification. That is to say, it is dependent on the number of degrees of freedom in the model. In the following figure the influence of the spatial and time parameters on the spatial discretization estimator can be seen more clearly. A drastic change in the accumulated error variation can be observed near the 40% of the load. This effect is caused by the plastification of one complete section in the models, i.e. a fast increase of the strains of the elements that belong to that section.

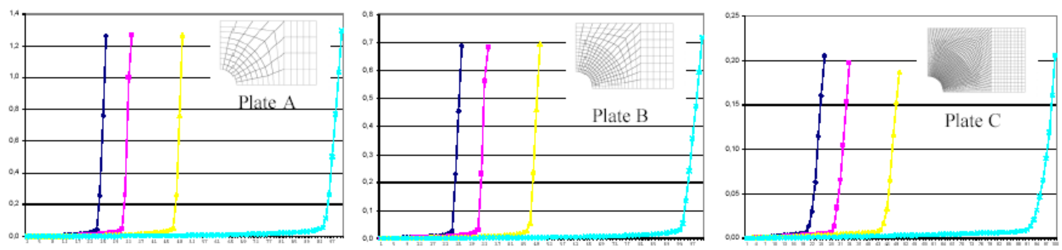

Figure 4. Spatial discretization error E ($\times 10^{-1}$ J) as a function of the number of load steps.

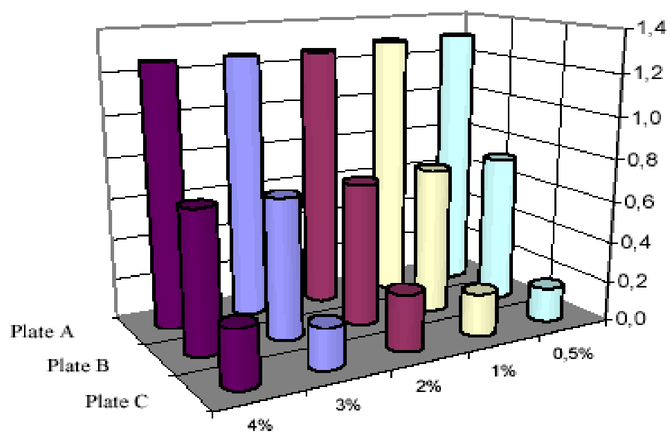

Figure 5. Spatial discretization error, E ($\times 10^{-1}$ J) – mesh and load increment influence.

In Figure (5) the maximum spatial error is drawn depending on the mesh and the load step. It is interesting to observe how the error estimator is not affected by the size of the load step but it is dependent on the element size.

4.2.2 Time error estimator

The results obtained using the time estimator are now discussed. In Figure 6 the accumulated error of the time discretization component is shown for every mesh and load increment. It is interesting to realize how the reduction of the load interval implies a decrease of the time discretization error component.

Figure 7, shows the time discretization error sequence with respect to the applied load interval. A quasi-linear variation of the time discretization error as a function of the load increment has been verified. In this way, if a linear regression ⌊ = k·.c (k constant) is done through the origin, the following solution can be presented (this regression is justified because the error is considered null when the increment tends to zero): based on an analysis with a load step quite large, the previous line equation can be calculated (the k parameter). Therefore a time error value for a suitable load interval can be predicted in order to obtain a rather small error or a user selected error.

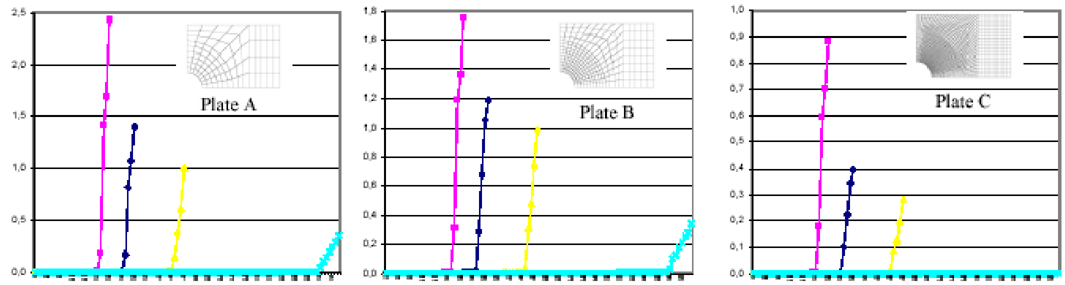

Figure 6. Time discretization error ξ ($\times 10^{-1}$ J) acumulated during the load process.

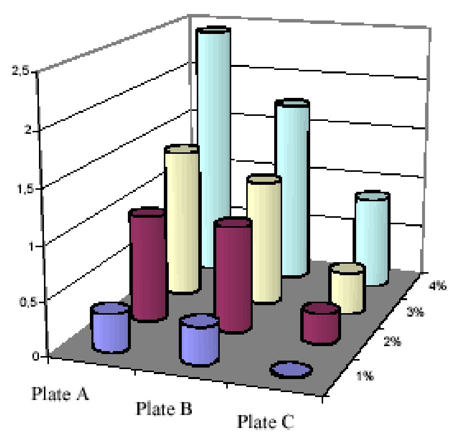

Figure 7. Time discretization error, ξ ($\times 10^{-1}$ J) – mesh and load increment influence.

4.3 Elastoplastic analysis – large strains

this time another elastoplastic analysis, now with large deformations, will be done. The applied load is σ =270 MPa. In Table 2 the maximum Von Mises stresses and plastic strains are detailed. The values of the plastic strains confirm the plastic state with large deformations.

Mesh	Maxim. displacement. (x 10^{-3} m)	Maxim. Von Mises stress. (x 10^5 Pa)	Maximum plastic strain. (%)
Plate A	4,99	9385	87,75
Plate B	5,04	9627	89,77
Plate C	5,05	9698	94,64

Table 2. Displacements - Stresses – Plastic strains.

4.3.1 Spatial error estimator

The considered load increments in these analyses with large strains are 2%, 1,5%, 1% and 0,5% of the total applied load. In Figure 8 the values of the spatial discretization error are shown for each mesh depending on the load increments. The spatial error is very little influenced by the load step size but it is very dependent on the element size. Figure 9 explains in detail the effect of the spatial variable and the load increment on the spatial discretization estimator.

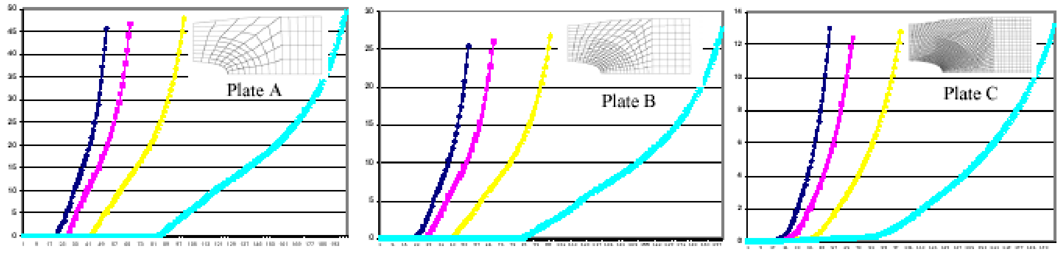

Figure 8. Spatial discretization error E ($\times 10^{-1}$ J) as a function of the number of load steps.

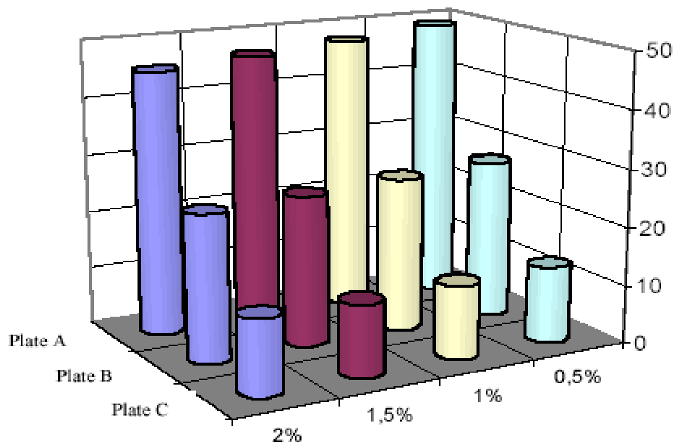

Figure 9. Spatial discretization error, E ($\times 10^{-1}$ J) – mesh and load increment influence.

In this case, as in the previous one, sudden changes in the curvature of the curves of Figure 8 can be detected. In the moment of each of these changes a vertical section of the model plastifies producing a very fast change in the strain values. The effect of the number of degrees of freedom and the load increment is shown in Figure 9. These values, obtained in the case of elastoplasticity with large strains, prove the little influence of the time variable on the spatial error estimator and the spatial error dependence upon the element size.

4.3.2 Time error estimator

The time discretization error component has been studied similarly to the small strain case. The results can be observed in Figure 10. The influence of the load increment variation on the time error component can be seen precisely. It is also possible to understand why this error is higher when the load increment increases. The time discretization error component increases proportionately to the load step size, as it can be seen in the figure.

Based on the curves of figure 11, it is established how, if the time error estimator proposed in this paper is used, the prediction methodology presented for the small strains case is also appropriate for a large strains elastoplastic analysis. Hence, an analysis made using a load step with a low computational cost permits the definition of the time discretization error for any other load step.

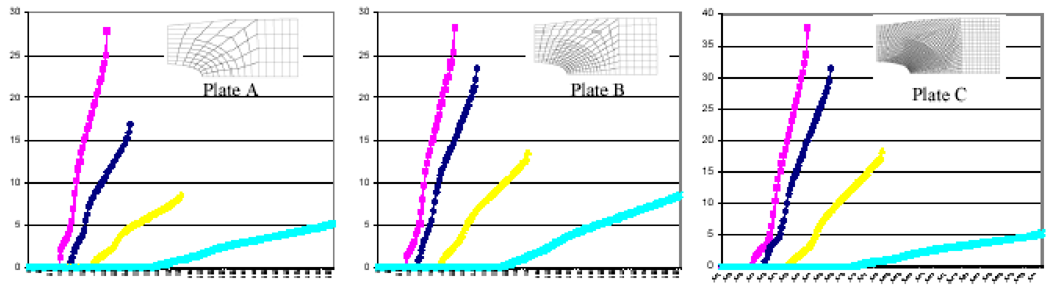

Figure 10. Time discretization error ξ (×10^{-1} J) during the load sequence.

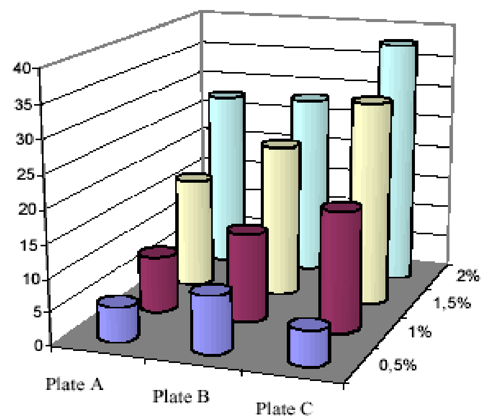

Figure 11. Time discretization error, ξ (×10^{-1} J) – mesh and load increment influence.

5.0 CONCLUSIONS

An examination of the previous results demonstrates how the spatial discretization error and the time discretization error depend on the spatial and time parameters respectively. Therefore the spatial estimator detects the element size variation and the time estimator reacts to a load increment change.

In relation with the spatial component, an estimator has been developed. This estimator evaluates only the error caused by the spatial discretization part and it is not affected by the time discretization error part. However, in the temporal case, the error estimator depends not only on the load increment, but also, with less influence, on the domain discretization. Despite this, it has been verified that the influence of the domain discretization is negligible if the load increment is quite low. The obtained results allow the prediction of the error for any load step using a simple linear regression. This knowledge allows the selection of the adequate increment in order to obtain a negligible time discretization error component compared with the spatial discretization error component, the most important component in large strain elastoplastic analysis. Finally, it has been verified that, in the elastoplastic case, the time error can be larger than the spatial error if an adequate load step size is used, i.e. not bigger than 1% of the total load.

A global error estimator is intended to develop in addition to these two error estimators in order to quantify together the spatial and the time discretization error. This global estimator will define the basis of an adaptive strategy to be applied in problems with large nonlinearities. The purpose is to study also the error component originated by the implicit method used to solve the equilibrium equations. This method is an iterative process that achieves only an approximate equilibrium of the equations on each load step.

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